UR-1421
ER-40685-870
hep-th/9505041



# Bi-Hamiltonian Structure of the Supersymmetric Nonlinear Schrödinger Equation

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## Abstract

We show that the supersymmetric nonlinear Schrödinger equation is a bi-Hamiltonian integrable system. We obtain the two Hamiltonian structures of the theory from the ones of the supersymmetric two boson hierarchy through a field redefinition. We also show how the two Hamiltonian structures of the supersymmetric KdV equation can be derived from a Hamiltonian reduction of the supersymmetric two boson hierarchy as well.

Bosonic integrable models have been studied in detail in the past [1-3]. These models have a very rich structure. However, only recently, after the discovery of the connection between hierarchies of integrable equations and discretized versions of the two-dimensional gravity [4], has there been a lot of interest in the high energy community in the study of these systems. This also has led to a renewed interest in the study of supersymmetric integrable systems for various reasons – the most important being the fact that a supersymmetric theory of gravity would be free from problems such as tachyonic states. However, a lot of properties of the supersymmetric integrable systems remain to be studied.

The first supersymmetric integrable system to be studied was the supersymmetric KP hierarchy (sKP) of Manin and Radul [5], which upon appropriate reduction, leads to the supersymmetric KdV equation (sKdV) (the first fermionic extension of KdV, though, is due to Kupershmidt [6]). The second Hamiltonian structure of this system was shown by Mathieu [7] to correspond to the superconformal algebra of the superstring theories. However, the bi-Hamiltonian nature of the system was not known until much later. The bi-Hamiltonian property is intimately connected with the integrability of a system and this aspect of sKdV was obtained [8-10] from a reduction of an even order sKP Lax operator. It was found that the simplest Hamiltonian structure of the KdV equation becomes a complicated nonlocal structure upon supersymmetrization.

Another interesting supersymmetric integrable system that has received a lot of attention, lately, is the supersymmetric nonlinear Schrödinger equation (sNLS) [11-12]. In [11] it was shown that the most general supersymmetric extension of the NLS is given by

$$\begin{aligned}\frac{\partial Q}{\partial t} &= -(D^4 Q) + 2\alpha(D\overline{Q})(DQ)Q - 2\gamma\overline{Q}Q(D^2Q) + 2(1-\alpha)\overline{Q}(DQ)^2\\ \frac{\partial \overline{Q}}{\partial t} &= (D^4\overline{Q}) - 2\alpha(DQ)(D\overline{Q})\overline{Q} + 2\gamma Q\overline{Q}(D^2\overline{Q}) - 2(1-\alpha)Q(D\overline{Q})^2\end{aligned} \tag{1}$$

where

$$\begin{aligned}Q &= \psi + \theta q\\ \overline{Q} &= \overline{\psi} + \theta\overline{q}\end{aligned} \tag{2}$$

are fermionic superfields which are complex conjugates of each other, and

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x} \tag{3}$$

defines the supercovariant derivative in the superspace with coordinates $z = (x, \theta)$ satisfying $D^2 = \partial$. However, it was shown in [12] that the standard tests of integrability hold for the system (1) only for

$$\alpha = -\gamma = 1 \tag{4}$$

In other words, the supersymmetric extension of NLS which is also integrable has the form,

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -(D^4 Q) + 2\left(D((DQ)\overline{Q})\right) Q \\ \frac{\partial \overline{Q}}{\partial t} &= (D^4 \overline{Q}) - 2\left(D((D\overline{Q})Q)\right) \overline{Q} \end{aligned} \tag{5}$$

Much like the sKdV, it was shown in [12] that the naive supersymmetrization of the Hamiltonian structures of the bosonic NLS equation goes through for the second Hamiltonian structure only for the particular values of the parameters in (4). However, a bi-Hamiltonian structure was still lacking. In this letter we will derive the first Hamiltonian structure of the system showing that the system is, indeed, bi-Hamiltonian for these values of the parameters. This will complete the analysis of the integrable structure of this system started in [12]. Like the sKdV system, the first structure, as we will show, will be extremely nonlocal.

In ref. [13], it was shown that the scalar Lax operator for the sNLS system can be identified with

$$\mathcal{L} = -\left(D^2 + \overline{Q}Q - \overline{Q}D^{-1}(DQ)\right) \tag{6}$$

where $D^{-1} = \partial^{-1}D$ is the pseudo super-differential operator and that the sNLS equations (5) can be written as the nonstandard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\mathcal{L}, (\mathcal{L}^2)_{\geq 1}\right] \tag{7}$$

In principle, given a Lax equation, the Hamiltonian structures of the theory can be derived from the Gelfand-Dikii brackets (appropriately extended to the superspace) [3,10,14]. However, eq. (7) is a nonstandard Lax equation and the definition of the Gelfand-Dikii brackets have so far been extended only for the bosonic systems in such cases [15]. The extension to superspace of such a generalization is technically much more involved and is

presently under study. However, here, we will follow an alternate approach and exploit the relation between the supersymmetric two boson system and the sNLS system to derive the Hamiltonian structures for the latter.

The integrable supersymmetric two boson equation (sTB) is given by [16]

$$\begin{aligned}\frac{\partial \Phi_0}{\partial t} &= -(D^4\Phi_0) + (D(D\Phi_0)^2) + 2(D^2\Phi_1) \\ \frac{\partial \Phi_1}{\partial t} &= (D^4\Phi_1) + 2(D^2((D\Phi_0)\Phi_1))\end{aligned} \tag{8}$$

where $\Phi_0$ and $\Phi_1$ are fermionic superfields. (We follow the notation in ref. [16].) This system can also be obtained from the following nonstandard Lax equation

$$\begin{aligned}L =& D^2 - (D\Phi_0) + D^{-1}\Phi_1 \\ \frac{\partial L}{\partial t} =& \left[L, (L^2)_{\geq 1}\right]\end{aligned} \tag{9}$$

It has already been shown that (8) is a tri-Hamiltonian system [16,17]

$$\partial_t \begin{pmatrix} \Phi_0 \\ \\ \Phi_1 \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta \Phi_0} \\ \\ \frac{\delta H_3}{\delta \Phi_1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta \Phi_0} \\ \\ \frac{\delta H_2}{\delta \Phi_1} \end{pmatrix} = \mathcal{D}_3 \begin{pmatrix} \frac{\delta H_1}{\delta \Phi_0} \\ \\ \frac{\delta H_1}{\delta \Phi_1} \end{pmatrix} \tag{10}$$

where we note that the first two Hamiltonian structures have the form

$$\mathcal{D}_1 = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix} \tag{11$a$}$$

$$\mathcal{D}_2 = \begin{pmatrix} -2D - 2D^{-1}\Phi_1 D^{-1} + D^{-1}(D^2\Phi_0)D^{-1} & D^3 - D(D\Phi_0) + D^{-1}\Phi_1 D \\ \\ -D^3 - (D\Phi_0)D - D\Phi_1 D^{-1} & -\Phi_1 D^2 - D^2\Phi_1 \end{pmatrix} \tag{11$b$}$$

The second Hamiltonian structure (the third as well) is highly nontrivial, but it is important to note that it has been checked, using superprolongation methods [18], that Jacobi identity holds for these structures [17]. The Hamiltonians of this system are given by

$$H_n = \frac{(-1)^{n+1}}{n} \mathrm{sTr}\, L^n = \frac{(-1)^{n+1}}{n} \int dz\, \mathrm{sRes}\, L^n \qquad n = 1, 2, \ldots \tag{12}$$

where "sRes" is the super residue which is defined to be the coefficient of the $D^{-1}$ term in the pseudo super-differential operator with $D^{-1}$ at the right. The first few charges have

the form

$$
\begin{aligned}
H_1 &= -\int dz\, \Phi_1 \\
H_2 &= -\int dz\, (D\Phi_0)\Phi_1 \\
H_3 &= \int dz\, \Big[(D^3\Phi_0) - (D\Phi_1) - (D\Phi_0)^2\Big]\Phi_1 \\
H_4 &= -\frac{1}{2}\int dz\, \Big[2(D^5\Phi_0) + 2(D\Phi_0)^3 + 6(D\Phi_0)(D\Phi_1) - 3\left(D^2(D\Phi_0)^2\right)\Big]\Phi_1
\end{aligned}
\tag{13}
$$

The crucial observation [16], for our analysis, is the fact that the sNLS equation and the sTB equation are related to each other through the field redefinition (Miura transformation)

$$\Phi_0 = -\,(D\ln(DQ)) + \left(D^{-1}(\overline{Q}Q)\right) \tag{14$a$}$$

$$\Phi_1 = -\overline{Q}(DQ) \tag{14$b$}$$

The field redefinitions (14) allows us to write the Lax operator (9) also as

$$L = G\widetilde{L}G^{-1} \tag{15}$$

where

$$
\begin{aligned}
G =&(DQ)^{-1} \\
\widetilde{L} =&D^2 - \overline{Q}Q - (DQ)D^{-1}\overline{Q}
\end{aligned}
\tag{16}
$$

We say that $L$ and $\widetilde{L}$ are gauge related. All this is very much like the bosonic case [19-22]. However, the important difference [13] is that it is not $\widetilde{L}$, rather its formal adjoint $\widetilde{L}^*$,

$$\widetilde{L}^* = -\left(D^2 + \overline{Q}Q - \overline{Q}D^{-1}(DQ)\right) = \mathcal{L} \tag{17}$$

which gives the sNLS equation as a nonstandard Lax equation (7). The relations in (14) are invertible and can be formally written as

$$Q = \left(D^{-1}\mathrm{e}^{\left(D^{-2}\left(-(D\Phi_0)+\Phi_1(L^{-1}\Phi_0)\right)\right)}\right) \tag{18$a$}$$

$$\overline{Q} = -\,\Phi_1\mathrm{e}^{\left(-D^{-2}\left(-(D\Phi_0)+\Phi_1(L^{-1}\Phi_0)\right)\right)} \tag{18$b$}$$

and these ((14) and (18)) define the connecting relation between the two theories.

Given these relations between the two theories and given the fact that we already know the Hamiltonian structures of the sTB system, we can obtain the Hamiltonian structures of the sNLS equations in the following way. Let us define the transformation matrix between the two systems as

$$P = \left[\frac{\delta(\Phi_0, \Phi_1)}{\delta(Q, \overline{Q})}\right] \tag{19}$$

where $\left[\frac{\delta(\Phi_0,\Phi_1)}{\delta(Q,\overline{Q})}\right]$ is the matrix formed from the Fréchet derivatives of $\Phi_0$ and $\Phi_1$ with respect to $Q$ and $\overline{Q}$. It follows now that the Hamiltonian structures of the two systems must be related as

$$\mathcal{D} = P\,\mathcal{D}^{\text{sNLS}}P^* \tag{20}$$

where $P^*$ is the formal adjoint of $P$ (with the matrix transposed). We note here the explicit structure of the transformation matrix for completeness.

$$P = \begin{pmatrix} -D(DQ)^{-1}D + D^{-1}\overline{Q} & -D^{-1}Q \\ -\overline{Q}D & -(DQ) \end{pmatrix} \tag{21}$$

It is useful to note that the matrix $P$ factorizes as

$$P = \widetilde{P}G \tag{22}$$

where

$$\widetilde{P} = \begin{pmatrix} -D^{-1} & -D^{-1}(DQ)^{-1}QD^2 \\ 0 & -D^2 \end{pmatrix} \tag{23$a$}$$

$$G = \begin{pmatrix} -\widetilde{L}^*(DQ)^{-1}D & 0 \\ D^{-2}\overline{Q}D & D^{-2}(DQ) \end{pmatrix} \tag{23$b$}$$

We also note from (20) that the Hamiltonian structures of the sNLS system can be obtained from those of the sTB system (written in terms of $Q$ and $\overline{Q}$) as

$$\mathcal{D}^{\text{sNLS}} = P^{-1}\mathcal{D}\,(P^*)^{-1} \tag{24}$$

where the inverse matrix has the form

$$P^{-1} = \begin{pmatrix} D^{-1}(DQ)(\widetilde{L}^*)^{-1}D & -D^{-1}(DQ)(\widetilde{L}^*)^{-1}Q(DQ)^{-1} \\ -\overline{Q}(\widetilde{L}^*)^{-1}D & -(1-\overline{Q}(\widetilde{L}^*)^{-1}Q)(DQ)^{-1} \end{pmatrix} \tag{25}$$

Armed with these relations, we note that if we use the second Hamiltonian structure of the sTB system, eq. (11b), we obtain from (24) that

$$\mathcal{D}_2^{\mathrm{sNLS}} = P^{-1}\mathcal{D}_2(P^*)^{-1} \tag{26}$$

which after using (25) and a lot of tedious algebra gives

$$\mathcal{D}_2^{\mathrm{sNLS}} = \begin{pmatrix} -QD^{-1}Q & -\frac{1}{2}D + QD^{-1}\overline{Q} \\ -\frac{1}{2}D + \overline{Q}D^{-1}Q & -\overline{Q}D^{-1}\overline{Q} \end{pmatrix} \tag{27}$$

This is, indeed, the correct second Hamiltonian structure that was derived in ref. [12] and provides a check on our method. (For those interested in rederiving this result, we note that it is enormously simpler to check that $\mathcal{D}_2$ factorizes as

$$\mathcal{D}_2 = P\,\mathcal{D}_2^{\mathrm{sNLS}}P^* \tag{28}$$

with $\mathcal{D}_2^{\mathrm{sNLS}}$ given in eq. (27).)

We are now in a position to derive the first Hamiltonian structure for the sNLS system. We note from (24) that

$$\mathcal{D}_1^{\mathrm{sNLS}} = P^{-1}\mathcal{D}_1(P^*)^{-1} \tag{29}$$

and using eqs. (11a) and (25), we obtain

$$\mathcal{D}_1^{\mathrm{sNLS}} = \begin{pmatrix} -D^{-1}(DQ)\Delta(DQ)D^{-1} & \begin{matrix} D^{-1}(DQ)\Delta\overline{Q} \\ +D^{-1}(DQ)(\widetilde{L}^*)^{-1}D^2(DQ)^{-1} \end{matrix} \\ & \\ \begin{matrix} \overline{Q}\Delta(DQ)D^{-1} \\ +(DQ)^{-1}D^2\widetilde{L}^{-1}(DQ)D^{-1} \end{matrix} & \begin{matrix} -\overline{Q}(\widetilde{L}^*)^{-1}D^2(DQ)^{-1} - (DQ)^{-1}D^2\widetilde{L}^{-1}\overline{Q} \\ -\overline{Q}\Delta\overline{Q} \end{matrix} \end{pmatrix} \tag{30}$$

where we have defined

$$\Delta \equiv (\widetilde{L}^*)^{-1}\left(D^2((DQ)^{-1}Q)\right)\widetilde{L}^{-1} \tag{31}$$

Like the first Hamiltonian structure of the sKdV equation, we note that this structure is highly nonlocal and, therefore, could not be obtained from a naive supersymmetrization

of the corresponding bosonic structure. It can now be checked explicitly that the sNLS equations (5) can be written in the Hamiltonian form

$$\partial_t \begin{pmatrix} Q \\ \\ \overline{Q} \end{pmatrix} = \mathcal{D}_1^{\mathrm{sNLS}} \begin{pmatrix} \frac{\delta H_3}{\delta Q} \\ \\ \frac{\delta H_3}{\delta \overline{Q}} \end{pmatrix} = \mathcal{D}_2^{\mathrm{sNLS}} \begin{pmatrix} \frac{\delta H_2}{\delta Q} \\ \\ \frac{\delta H_2}{\delta \overline{Q}} \end{pmatrix} \tag{32}$$

where the Hamiltonians can be obtained from (13) (and are defined in ref. [13]). (We note here that it is rather involved to check (32) for the first structure in (30). It is much easier, however, to check that

$$\begin{pmatrix} \frac{\delta H_3}{\delta Q} \\ \\ \frac{\delta H_3}{\delta \overline{Q}} \end{pmatrix} = \left(\mathcal{D}_1^{\mathrm{sNLS}}\right)^{-1} \begin{pmatrix} -(D^4 Q) + 2\left(D((DQ)\overline{Q})\right) Q \\ \\ (D^4 \overline{Q}) - 2\left(D((D\overline{Q})Q)\right) \overline{Q} \end{pmatrix} \tag{33}$$

and the two are equivalent.) The two Hamiltonian structures satisfy the Jacobi identity since the Hamiltonian structures of the sTB system do and define a recursion operator which would relate all the Hamiltonian structures as well as the conserved quantities of the system in a standard manner. However, as is clear from the structure of the first Hamiltonian structure, it is extremely nontrivial. Our result, therefore, shows that the sNLS system (5) is a bi-Hamiltonian system and completes the analysis of the integrability structure of this theory.

To conclude, we will now indicate how the Hamiltonian structures of the sKdV equation can be derived from those of the sTB system through a reduction. We note that the sKdV [16] can be embedded into the sTB system in the following manner. Let us look at the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = [L, (L^3)_{\geq 1}] \tag{34}$$

with $L$ defined in (9). A simple calculation gives

$$(L^3)_{\geq 1} = D^6 + 3D\Phi_1 D^2 - 3D^2(D\Phi_0)D^2 + 3(D\Phi_0)^2 D^2 + 6\Phi_1(D\Phi_0)D \tag{35}$$

which leads to the dynamical equations (from (34))

$$\frac{\partial \Phi_1}{\partial t} = -\,(D^6\Phi_1) - 3D^2\Big(\Phi_1(D\Phi_0)^2 + (D^2\Phi_1)(D\Phi_0) + \Phi_1(D\Phi_1)\Big) \tag{36$a$}$$

$$\frac{\partial \Phi_0}{\partial t} = -(D^6\Phi_0) + 3D\Big(\Phi_1(D^2\Phi_0) - 2(D\Phi_1)(D\Phi_0) - \frac{1}{3}(D\Phi_0)^3 + (D\Phi_0)(D^3\Phi_0)\Big) \tag{36$b$}$$

We immediately see that the identification

$$\begin{aligned}\Phi_0 &= 0\\ \Phi_1 &= \Phi\end{aligned} \tag{37}$$

gives the sKdV equation

$$\frac{\partial \Phi}{\partial t} = -(D^6\Phi) - 3D^2\left(\Phi(D\Phi)\right) \tag{38}$$

and shows how the sKdV equation is embedded in the sTB system as a nonstandard Lax equation (34) with the Lax operator

$$L = D^2 + D^{-1}\Phi \tag{39}$$

It is easy to see from (13) that with the condition in (37), the even Hamiltonians vanish whereas the odd ones are the same as those for the sKdV system.

The reduction in (37) imposes a constraint on the system. Consequently, the Hamiltonian structures of the sKdV system can be obtained from those of the sTB system through a Dirac procedure [23] as follows. Let $\mathcal{D}$ be one of the Hamiltonian structures of the sTB system and $H$ denote one of the odd Hamiltonians of the system. Then, we have

$$\partial_t \begin{pmatrix} \Phi_0 \\ \\ \Phi_1 \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H}{\delta \Phi_0} \\ \\ \frac{\delta H}{\delta \Phi_1} \end{pmatrix} \tag{40}$$

If we now use eq. (37) in (40), we obtain

$$\partial_t \begin{pmatrix} 0 \\ \\ \Phi \end{pmatrix} = \begin{pmatrix} \overline{\mathcal{D}}_{11} & \overline{\mathcal{D}}_{12} \\ \overline{\mathcal{D}}_{21} & \overline{\mathcal{D}}_{22} \end{pmatrix} \begin{pmatrix} \overline{\frac{\delta H}{\delta \Phi_0}} = v_0 \\ \\ \overline{\frac{\delta H}{\delta \Phi_1}} = v_1 \end{pmatrix} \tag{41}$$

where $\overline{\mathcal{O}}$ denotes the quantities $\mathcal{O}$ calculated with $\Phi_0 = 0$ and $\Phi_1 = \Phi$. From (41), we immediately see that consistency requires

$$v_0 = -\overline{\mathcal{D}}_{11}^{-1}\overline{\mathcal{D}}_{12} v_1 \tag{42}$$

and that we can write

$$\partial_t \Phi = \overline{\mathcal{D}}_{21} v_0 + \overline{\mathcal{D}}_{22} v_1 = \mathcal{D}^{\mathrm{sKdV}} v_1 \tag{43}$$

with ($v_1$ for odd Hamiltonians is the same as $\frac{\delta H}{\delta \Phi}$ for the sKdV system.)

$$\mathcal{D}^{\mathrm{sKdV}} = \overline{\mathcal{D}}_{22} - \overline{\mathcal{D}}_{21}\overline{\mathcal{D}}_{11}^{-1}\overline{\mathcal{D}}_{12} \tag{44}$$

Using equation (11b), we note that

$$(\overline{\mathcal{D}}_2)_{11}^{-1} = -\frac{1}{2}D(D^3+\Phi)^{-1}D \tag{45}$$

and using (44) we readily obtain the standard second Hamiltonian structure of sKdV

$$\mathcal{D}_2^{\mathrm{sKdV}} = -\frac{1}{2}(D^5 + 3\Phi D^2 + (D\Phi)D + 2(D^2\Phi)) \tag{46}$$

The first Hamiltonian structure can also be obtained in a simple manner through the Dirac reduction. However, we should remember that since, in the limit $\Phi_0 = 0$, all the even charges in (13) vanish, the first Hamiltonian structure of the sKdV is obtained from $\mathcal{D}_0$ (and not from $\mathcal{D}_1$, as would be naively expected) given by

$$\mathcal{D}_0 = R^{-1}\mathcal{D}_1 \tag{47}$$

where $R = \mathcal{D}_2\mathcal{D}_1^{-1}$ is the recursion operator of the sTB system. It is easy to show that

$$\overline{R}^{-1} = \begin{pmatrix} 4D\left(\mathcal{D}_2^{\mathrm{sKdV}}\right)^{-1}D & -2D\left(\mathcal{D}_2^{\mathrm{sKdV}}\right)^{-1}D^3 \\ & \\ 2D^3\left(\mathcal{D}_2^{\mathrm{sKdV}}\right)^{-1}D & 2D^2(D^3+\Phi)^{-1}D^{-1}(\Phi D^2 + D^2\Phi)\left(\mathcal{D}_2^{\mathrm{sKdV}}\right)^{-1}D^3 \end{pmatrix} \tag{48}$$

Once again, using the Dirac reduction relation (44), we obtain

$$\mathcal{D}_1^{\mathrm{sKdV}} = (\overline{\mathcal{D}}_0)_{22} - (\overline{\mathcal{D}}_0)_{21}(\overline{\mathcal{D}}_0)_{11}^{-1}(\overline{\mathcal{D}}_0)_{12} \tag{49}$$

This gives

$$\mathcal{D}_1^{\mathrm{sKdV}} = -2D^2(D^3+\Phi)^{-1}D^2 \tag{50}$$

which is the nonlocal structure for the sKdV obtained in refs. [8-10]. Our derivation, however, shows that this structure satisfies the Jacobi identity since the Hamiltonian structures of sTB do. (The Jacobi identity for the first Hamiltonian structure, to the best of our knowledge, has not yet been demonstrated.)

To conclude, we have derived in this letter, the bi-Hamiltonian structures of the sNLS system starting from those of the sTB system. This completes the analysis of the integrability structure of the sNLS system. The derivation of these structures as Gelfand-Dikii brackets remains an open question and is presently under study. We have also shown how the Hamiltonian structures of the sKdV system can be obtained from those of the sTB system through a Dirac reduction. This provides an indirect proof of the Jacobi identity for these structures.

**Acknowledgements**

This work was supported in part by the U.S. Department of Energy Grant No. DE-FG-02-91ER40685. J.C.B. would like to thank CNPq, Brazil, for financial support.

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